# Design of a Rocket Nozzle's Divergent Section by means of a Bézier Curve Approximation

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***Abstract*—Rockets depend on the design of convergent-divergent nozzles in order to obtain their thrust. This document focuses on the design methodology of bell nozzles by use of a bézier curve for the definition of the divergent cone. It also demonstrates that pure algebraic parabolas are not consistent with the requirements set by the literature in the design of bell nozzles.**

***Index Terms*—Propulsion, Nozzles, Bell nozzle, Bézier Curve**

## NOMENCLATURE

$\eta$: Nozzle length ratio
$A$: Transversal Area ($m^2$)

### A. *Subscripts*

$t$: throat conditions
$i$: Inflexion Point I conditions
$e$: Escape Point E conditions
$Q$: Intersection point Q

## I. INTRODUCTION

The design of rocket nozzles has been discussed in multiple sources in the literature. The first comprehensive view of the subject was performed by G.V.R. Rao of the National Engineering Science Co.

Rao describes the design of a rocket nozzle via the method of characteristics. [1] However, due to the technological impairments at the time of publication, he proposes a simplified method for the design of a bell nozzle by use of a "parabola". [2]

This parabola is in actuality the geometric construction of a parabola-like curve between two points, and tangent to specific lines passing through said points. This geometric construction can be described algebraically by means of a quadratic Bézier curve.

Further proof of the "non conformity" of a true algebraic parabola to the design requirements set forth by the method will be presented at the end of this article.

## II. METHODOLOGY

This design methodology presupposes an isentropic expansion analysis, and therefore, that the throat and escape radius of the nozzle are already known.[1]

The throat region of the nozzle is defined by both the throat radius and a pair of tangent circular arcs. The first arc is tangent to the convergent section of the nozzle, and tangent to the second arc. Rao describes a radius for said arcs of $1.5r_t$ and $0.4r_t$ respectively. [1] Figure 1 presents an adequate description of the throat and divergent bell section of the nozzle. The divergent arc is tangent to the convergent arc at the throat (point t), and ends at the inflexion point I (Maximum angle of divergent nozzle wall)

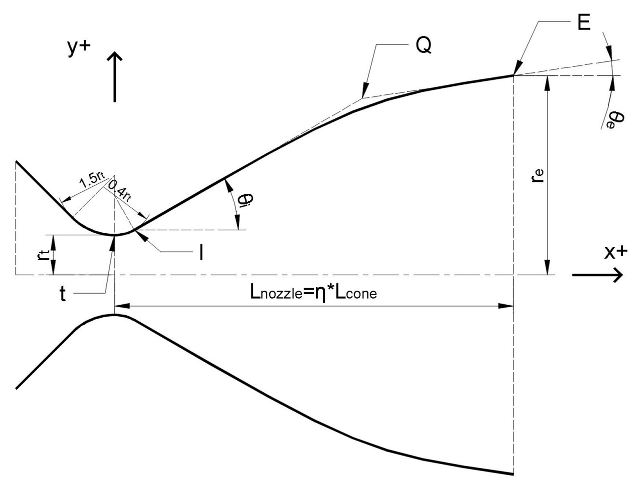


Fig. 1. Design of a rocket nozzle. Source: Own work based on Rocket Propulsion Elements

As can be seen from figure 1, there is a quadratic Bézier curve between points I and E. The determination of said curve shall be explored further in following sections. Additionally, there are specific angles defined for the nozzle walls at points I and E. These angles are necessary for the definition of the curve, to ensure tangency with the divergent arc, and to ensure the closest approximation to a completely axial flow at the nozzle exit.

The length of the parabolic nozzle is determined by two main factors, the reference length and the length ratio between the final length and said reference length. The reference length is defined as the length of an equivalent $15°$ nozzle. [3]

The length ratio ($\eta$) is essentially the percentage of length the bell nozzle will be with respect to an equivalent $15°$ nozzle. It is generally between the values of 60% and 100%. However, the most commonly used value is that of the 80% bell nozzle.

The design procedure has the following steps:

1) Starting from the throat and escape radius (determined via an isentropic expansion), determine the divergent expansion area ratio ($\epsilon$) for the nozzle

[1] If said analysis has not been done, the book *Rocket Propulsion Elements* (Chapter 3) presents a good starting point. [3]

2) Select a nozzle length ratio
3) Identify the initial and final angle for the given length ratio and expansion area ratio.
4) From said angles, determine the nozzle wall slopes at the inflexion point I and escape point E.
5) Determine the x and y coordinates for the reference points I and E.
6) Determine the intercepts with the y axis for the tangent lines to the nozzle wall at points I and E (IQ and QE).
7) Determine the intersection point between said lines (Point Q).
8) Apply the Quadratic Bézier Curve method to define the final segment of the nozzle wall.

The expansion area ratio in the first step can be determined by this formula:

$$\epsilon = \frac{A_e}{A_t} \tag{1}$$

The recommended length ratio is 0.8 (80%) for most nozzle contours.

## III. Determination of Reference Angles

The book *Rocket Propulsion Elements* presents the graph shown in figure 2 as the main method for finding the initial and final angles starting from the expansion area ratio and the length ratio. [3]

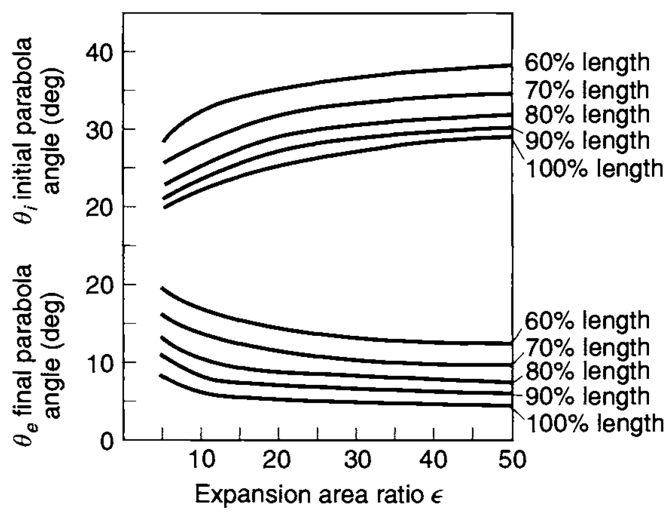


Fig. 2. Parabola Angles for Bell Nozzle. Source: *Rocket Propulsion Elements*, Sutton & Biblarz

Although the graph presented may be sufficient for some designs, a more general approach may be needed. To solve that problem a series of points were obtained for each curve and a polynomial regression analysis was carried out. This analysis gave a polynomial expression for each length ratio.

### A. Length ratio: 60%

$$\theta_{i-60\%} = 0.000220\epsilon^3 - 0.02424\epsilon^2 + 0.9341\epsilon + 24.88$$

$$\theta_{e-60\%} = -0.000142\epsilon^3 + 0.01667\epsilon^2 - 0.6714\epsilon + 22.39$$

### B. Length ratio: 70%

$$\theta_{i-70\%} = 0.000137\epsilon^3 - 0.0175\epsilon^2 + 0.7921\epsilon + 21.86$$

$$\theta_{e-70\%} = -0.000134\epsilon^3 + 0.01535\epsilon^2 - 0.6133\epsilon + 18.87$$

### C. Length ratio: 80%

$$\theta_{i-80\%} = 0.000142\epsilon^3 - 0.01789\epsilon^2 + 0.8015\epsilon + 19.08$$

$$\theta_{e-80\%} = -0.000162\epsilon^3 + 0.01683\epsilon^2 - 0.5981\epsilon + 15.52$$

### D. Length ratio: 90%

$$\theta_{i-90\%} = 0.000125\epsilon^3 - 0.01648\epsilon^2 + 0.7688\epsilon + 17.63$$

1) *$\epsilon$: 0-10:*

$$\theta_{e-90\%} = 0.000727\epsilon^3 + 0.0189\epsilon^2 - 0.9315\epsilon + 15.33$$

2) *$\epsilon$: 10-20:*

$$\theta_{e-90\%} = -0.00211\epsilon^3 + 0.1101\epsilon^2 - 1.948\epsilon + 19.18$$

3) *$\epsilon$: 20-50:*

$$\theta_{e-90\%} = 0.0000125\epsilon^3 - 0.00156\epsilon^2 + 0.0238\epsilon + 7.43$$

### E. Length ratio: 100%

$$\theta_{i-100\%} = 0.000070\epsilon^3 - 0.01084\epsilon^2 + 0.6094\epsilon + 17.27$$

1) *$\epsilon$: 0-10.5:*

$$\theta_{e-100\%} = -0.0002289\epsilon^3 + 0.0275\epsilon^2 - 0.7996\epsilon + 11.94$$

2) *$\epsilon$: 10.5-23.3:*

$$\theta_{e-100\%} = -0.000463\epsilon^3 + 0.288\epsilon^2 - 0.6287\epsilon + 10.26$$

3) *$\epsilon$: 23.3-50:*

$$\theta_{e-100\%} = 0.0000412\epsilon^3 - 0.00442\epsilon^2 + 0.122\epsilon + 4.41$$

### F. Final slopes

The reference angles are obtained in order to find the slopes of the tangent lines at the inflexion and escape points. Said slopes can be obtained via the following equations:

$$m_i = \tan(\theta_i) \tag{2}$$

$$m_e = \tan(\theta_e) \tag{3}$$

## IV. Determination of Reference Points

### A. *Inflection Point*

The Inflection Point corresponds to the intersection of the divergent circular arc and the bézier curve. This point's x and y coordinates can be determined by the geometric constraints set by the literature.

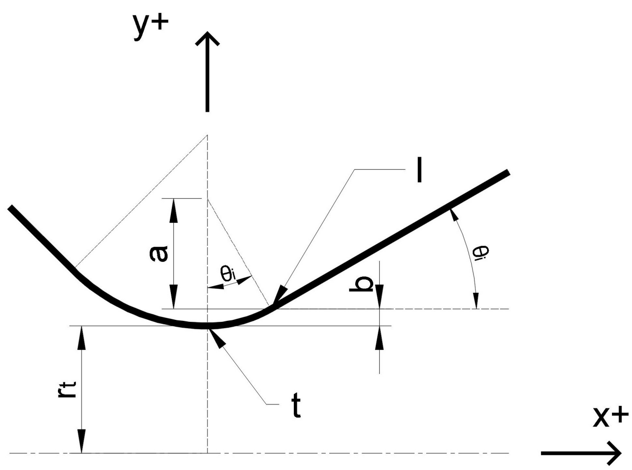


Fig. 3. Nozzle Inflection Point. Source: Own work

Figure 3 is a closer inspection of figure 1 with respect to the throat area. From this figure, equations 4 and 5 can be derived.

$$x_i = (0.4 * r_t) * \sin(\theta_i) \tag{4}$$

$$a = (0.4 * r_t) * \cos(\theta_i)$$

$$b = (0.4 * r_t) - a$$

$$y_i = r_t * [1 + 0.4 * (1 - \cos(\theta_i))] \tag{5}$$

### B. *Escape Point*

The Escape Point refers to the end of the Bézier curve that also corresponds to the end of the divergent section. This values are generally considered defined by the initial conditions of the problem.

$$x_e = \eta * \left( \frac{r_e - r_t}{\tan(15°)} \right) \tag{6}$$

Where:

- $\eta$: Ratio of the length of the parabolic nozzle to the length of the reference conic nozzle with $15°$ semi-angle.

$$y_e = r_e \tag{7}$$

### C. *Line Intercepts*

As a part of the bézier method, it is necessary to define the lines that pass through the points I and E with the slopes $m_i$ and $m_e$ respectively. Essentially, it is defining the lines IQ and QE as seen on figure 1.

$$y_i = m_i * x_i + b_i$$

$$y_e = m_e * x_e + b_e$$

Solving for the line intercept value b, the following expressions can be found.

$$b_i = y_i - m_i * x_i \tag{8}$$

$$b_e = y_e - m_e * x_e \tag{9}$$

### D. *Intermediate Point*

The following step is to calculate the x and y coordinate for the Q point by calculating the intersection point of lines IQ and QE. This analysis yields the following equations.

$$x_Q = \frac{b_e - b_i}{m_i - m_e} \tag{10}$$

$$y_Q = m_i x_Q + b_i \tag{11}$$

## V. Bézier Curves

Previously it was discussed that a simple algebraic parabola is not a good fit to describe the nozzle wall while retaining all of the conditions set forth of points position and entry and exit slopes. The reasons for this will be discussed further in the following section. The main issue to deal with becomes, how can the nozzle wall be described mathematically if not by an algebraic parabola?

Rao describes a method of geometric construction to create a "geometric parabola" that satisfies the required constraints. This method can be described in a purely algebraic sense by use of a Bézier quadratic curve.

A Bézier curve is a parametric function dependent on a series of control points. Bézier curves can be found from order n=1 (linear curves) up to higher orders. [4] For the uses in nozzle design we will focus on second order (quadratic) curves.

Since Bézier curves are parametric functions, they depend on a general variable, in this case t, to describe values for the x and the y coordinate, instead of relating a value of y and x directly as in a normal function. By definition, t is set to have values that range from 0 to 1.

The curve can be explained by a series linear interpolation between a series of control points. For a Bézier curve of order 1 (linear), two points are needed to define the curve. A point $B_{linear}$ can be defined for said curve as the interpolation between points $P_0$ and $P_1$, defining a single line between both points, with t varying from 0 to 1.

$$B_{linear}(t) = (1-t)P_0 + tP_1 \tag{12}$$

This linear bézier function describes the curve by a series of linear interpolations between the points $P_0$ and $P_1$ for varying values of t. Each value of t gives a different value of interpolation along the line between points $P_0$ and $P_1$. Figure 4 gives a representation of this process. The bold line in the figure gives the drawn line from point $P_0$ ($t = 0$) up to the value of the interpolation for $t = 0.4$.

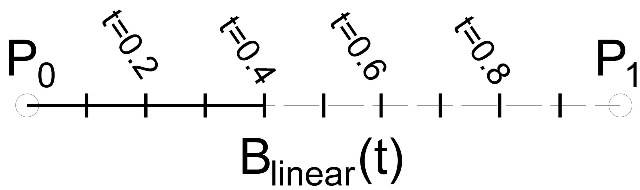


Fig. 4. Linear Bézier method. Source: Own work

For a second order curve, with control points $P_0$, $P_1$, and $P_2$, a similar approach of linear interpolations can be applied. For a given value of t between 0 and 1, a first interpolation can be done between point $P_0$ and $P_1$, to yield a first point on the line $P_0P_1$. A second point can be found out through interpolation for the same value of t along the line $P_1P_2$. If a third interpolation is done, again for the same value of t, between the two points already found, this third point will be a point along a bézier curve $B(t)$. Figure 5 presents a visual representation of the process.

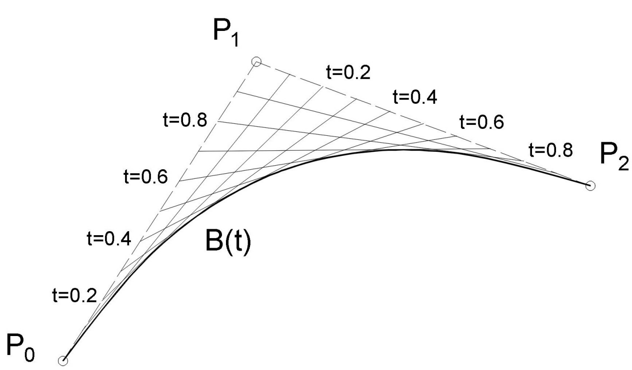


Fig. 5. Quadratic Bézier method. Source: Own work

Equation 13 gives the algebraic expression of the curve with respect to the variable t.

$$B(t) = (1-t)^2 P_0 + 2(1-t)tP_1 + t^2 P_2 \tag{13}$$

Taking equation 13, and applying it to the nozzle shown in Figure 1, the expression can be modified as follows.

$$B(t) = (1-t)^2 I + 2(1-t)tQ + t^2 E \tag{14}$$

Using equation 14 has a reference, the equation can be decomposed by use of the Cartesian coordinate for each point.

$$B_x(t) = (1-t)^2 x_i + 2(1-t)tx_Q + t^2 x_e \tag{15}$$

$$B_y(t) = (1-t)^2 y_i + 2(1-t)ty_Q + t^2 y_e \tag{16}$$

This curve passes by both points I and E, and respects the slopes at said points.

## VI. The algebraic parabola problem

The initial alternative of using an algebraic expression of a parabola is tempting. This is mainly due to the easiness of graphing a parabolic curve, and the intuitive nature of the function. This however presents multiple problems. In order to present said problems an attempt to describe the nozzle wall by means of a parabola will be made.

A parabola can be described by the following expression.

$$y = ax^2 + bx + c \tag{17}$$

Because of how we defined the coordinate system in figure 1, we will asume a parabola defined in the y coordinate system.

$$x = ay^2 + by + c \tag{18}$$

The derivative of said equation will be as follows.

$$x' = 2ay + b \tag{19}$$

Additionally, due to the change in the coordinate system, there is a need to redefine the slope of the wall at points I and E.

$$m_i = \tan(90^\circ - \theta_i) \tag{20}$$

$$m_e = \tan(90^\circ - \theta_e) \tag{21}$$

Using figure 1 as a reference the following requirements must be met.

1) Function slope must be equal to $m_i$ in point I
2) Function slope must be equal to $m_e$ in point E
3) The function must contain point I
4) The function must contain point E

For the case of the parabola, this requirements can be expressed by the following equations.

$$2ay_i + b = m_i \tag{22}$$

$$2ay_e + b = m_e \tag{23}$$

$$ay_i^2 + by_i + c = x_i \tag{24}$$

$$ay_e^2 + by_e + c = x_e \tag{25}$$

There exist 3 unknowns in the form of a, b, and c and four algebraic expressions, which means the system can be solved. From equations 22 and 23, the values of a and b can be obtained.

$$a = \frac{m_i - m_e}{2(y_i - y_e)} \tag{26}$$

$$b = \frac{y_i m_e - y_e m_i}{y_i - y_e} \tag{27}$$

From equations 24 and 25, two variants of c can be found.

$$c = x_i - ay_i^2 - by_i \tag{28}$$

$$c = x_e - ay_e^2 - by_e \tag{29}$$

For the algebraic parabola to work, equations 28 and 29 must yield the same result. They do not yield the same result for most cases. Several scenarios where analyzed for different expansion ratios without success. The only situation where both equations coincide is with an expansion ratio of 1.05 for an 80% nozzle.

Additionally depending on the value of c used (eq 28 or 29), the parabola does not pass by point I or point E.

A similar exercise was carried out assuming an axis centered parabola ($b = 0$) which resulted in a better adjustment at points I and E. But this solution does not respect the slopes of the nozzle wall at those points.

## VII. Conclusions

The general design of a bell nozzle presents certain advantages and disadvantages to the designer. The contour of the nozzle wall permits an increment, on average of around 1% in performance for most cases. [3] This design also allows a general reduction on the material used on the nozzle, which leads to lower weight of the rocket. The manufacturing of this design however presents the need of more complicated processes that must be considered.

The design methodology presented is expected to provide a more easier iteration method mainly for the easiness of it's application within a computer application, in contrast to the more complex geometric approach.

It is important to reiterate the non compliance of the algebraic parabola with the problem restrictions presented by the literature. Even if the parabola complies with the original points I and E, there may be noncompliance with the established wall slopes.

It should also be noted that this method remains an approximation of the ideal nozzle contour used for the optimum gas expansion within the nozzle wall. The method of characteristics remains as the ideal method of nozzle wall design.

## References

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